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Quasi Risk-Neutral Pricing
in Insurance

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Motivation

- explain and help to overcome the **discrepancy** of pricing principles discussed in insurance economics as compared to “ad hoc” pricing methods employed in practical insurance
- economic interpretation of the **distortion approach** discussed intensively in literature over the past decade

Pricing Axioms

1. Non-negative risk loading and no rip-off
2. Monotonicity with respect to first stochastic dominance order
3. Subadditivity
4. Additivity for comonotonic risks
5. Continuity

Price representation

$$H(X) = \int_{(a,b)} x \underbrace{\pi'(F_X(x)) f_X(X)}_{f_X^*(X)} dx,$$

where $0 \leq a < b \leq \infty$ and

(i) $\pi \in [0, 1]$, $\pi[0] = 0$, $\pi[1] = 1$

(ii) $\pi' \geq 0$

(iii) $\pi'' \geq 0$

Examples of π

- Dual power transform
- Proportional hazard transform
- Normal (Wang) transform
- Beta transform
- etc.

Problem

- the class of distortion operators which possess properties (i) – (iii) is rich
- even once a distortion function has been found preferable over the others, the **problem of calibration** still needs to be solved
- ➔ lack of a uniform pricing rule does not help pricing actuaries to become comfortable with distortion approach

Proposed solution in 2-step process

1. motivate a **uniform** choice for π based on relative entropy minimization
2. analyze under which circumstances the calibration is **unique** under **risk-neutral pricing**

Step one

Minimize the **relative entropy** of the transformed probability measure $\pi(F)$ with respect to the best estimate distribution F , i.e.

$$\delta \left[- \int_{(0,1)} f(p) \ln(f(p)) dx - \lambda \left(\int_{0,1} p f(p) dp - \xi \right) \right] = 0,$$

s.t.

(i) – (iii)

Step one - cont

from which follows that

$$f(p) = -\frac{\lambda e^{\lambda p}}{1 - e^{\lambda}}, p \in [0, 1] \text{ and } \lambda \geq 0$$

which implies with $\pi' := f$

$$\pi(p) = \frac{1 - e^{\lambda p}}{1 - e^{\lambda}}$$

Step 2

Lemma

For any two **comonotonic** and **partially Lorenz ordered** prospects X_1 and X_2 with **identical** and **finite expectation** such that $X_1 \leq_L X_2$ it holds true that

$$\begin{aligned} \mathcal{E}(u(-X_1)) \geq \mathcal{E}(u(-X_2)) &\Leftrightarrow H(X_1) \leq H(X_2) \\ \Leftrightarrow \mathcal{E}(u(-X_1)) \geq \mathcal{E}(u(-X_2)) &\Leftrightarrow \mathcal{E}^*(X_2) \leq \mathcal{E}^*(X_2), \end{aligned}$$

for **any** increasing concave risk utility function u .

Lorenz order

the j -th moment distribution

$$F_X^{(j)}(x) = \frac{\int_0^x y^j dF(y)}{\int_0^\infty y^j dF(y)}$$

Lorenz order - cont

First stochastic dominance order

$$X_1 \leq_{st} X_2 \Leftrightarrow F_{X_1}^{(0)}(x) \geq F_{X_2}^{(0)}(x)$$

Lorenz order

$$X_1 \leq_L X_2 \Leftrightarrow F_{X_1}^{(1)}(x) \geq F_{X_2}^{(1)}(x)$$

Step 2 - cont

Subject to the Lemma, with $H(X) \equiv H_{\pi_\lambda}(X)$, we have

$$H_{\lambda_1}(X_1) \leq H_{\lambda_2}(X_2),$$

if and only if

$$\frac{e^{\lambda_1} + \lambda_1 e^{\lambda_1} + 1}{\lambda_1(e^{\lambda_1} - 1)} = \xi_1 \quad \text{and} \quad \frac{e^{\lambda_2} + \lambda_2 e^{\lambda_2} + 1}{\lambda_2(e^{\lambda_2} - 1)} = \xi_2$$

Step 2 - cont

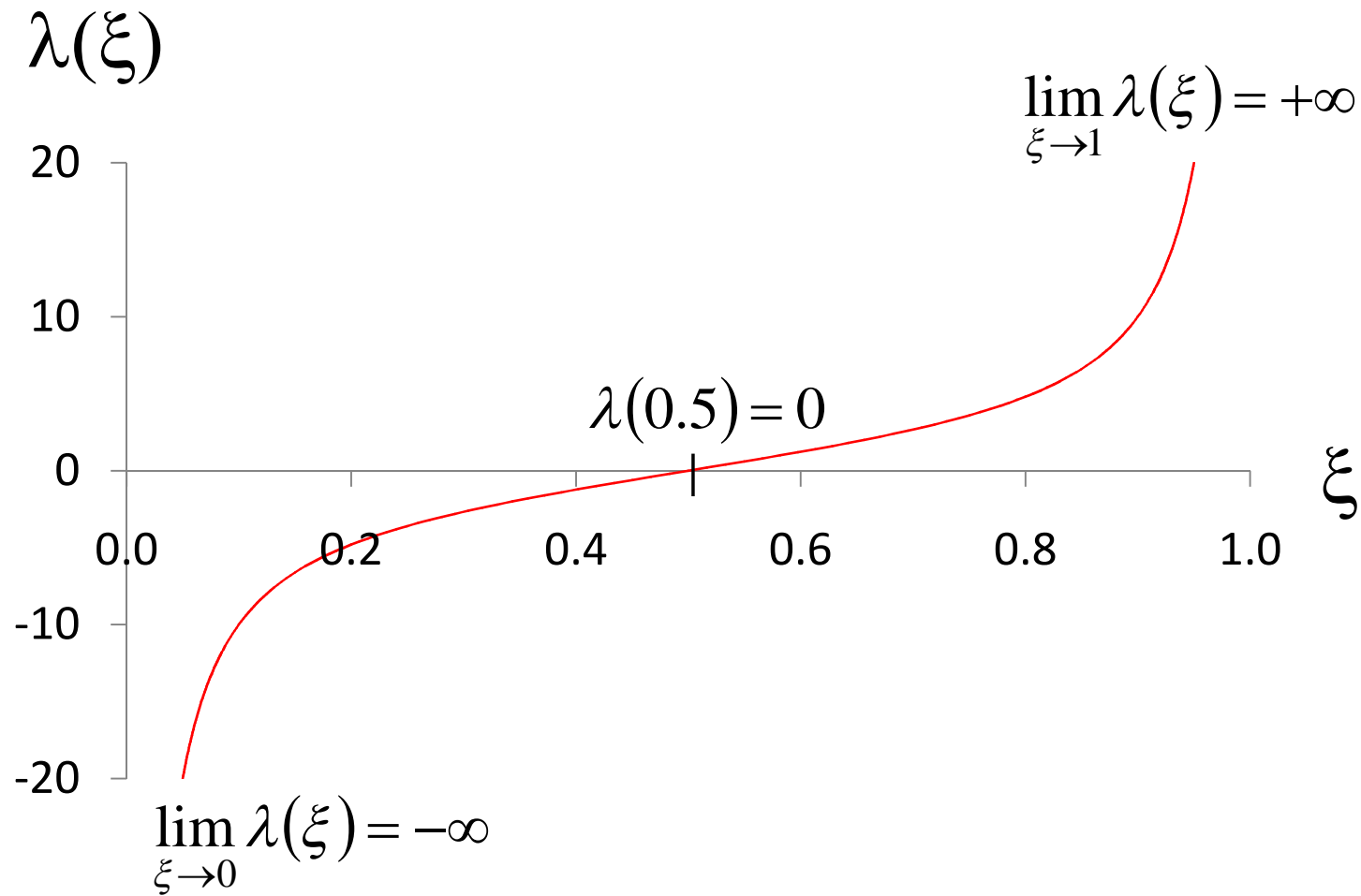
where ξ_1 and ξ_2 are the arguments for which the respective Lorenz functions of X_1 and X_2 take on their mean value.

Interpretation:

Under the prerequisites of the Lemma the market is

- *quasi risk-neutral* under the relative entropy minimizing measure $F^* := \pi_\lambda(F)$,
- the limit case of risk-neutrality being obtained for $\lambda=0$ implying $F^* = F$

Graph of $\lambda(\xi)$



Conclusions

- the exponential distortion operator is the **optimal** choice under relative entropy minimization
- the calibration of the risk-aversion parameter λ is **unique** if with respect to a set of random prospects the market is quasi risk-neutral

Beyond the talk

- a practical example where prices and Risk Adjusted Capital (RAC) values are calculated and interpreted
- validity of the quasi risk-neutral pricing framework presented in a context where the prerequisites of the Lemma are not given, i.e. if there is violation with respect to
 - comonotonicity and/or
 - Lorenz order

Summary statistics of case study

LOC	Ins Val (10 ⁶)	MPL (10 ⁶)	c	$\mathcal{E}(Z_i)$ (10 ³)	H(Z _i) (10 ³)	RAC (10 ⁶)	RAC* (%)	RAC ⁺ (%)
1	261.9	26.6	4.31	21.8	49.4	1.38	21.4	25.0
2	233.7	19.3	4.26	16.5	37.3	1.04	16.3	18.1
3	130.2	13.2	4.17	12.4	27.8	0.77	12.0	12.4
4	64.1	8.8	4.08	9.1	20.4	0.57	8.9	8.2
5	47.8	7.9	4.05	8.4	18.7	0.52	8.1	7.5
6	36.7	7.3	4.04	7.9	17.6	0.49	7.7	6.9
7	29.9	6.9	4.03	7.5	16.7	0.46	7.2	6.5
8	22.8	6.4	4.00	7.1	15.7	0.43	6.7	6.0
9	19.3	5.7	3.96	6.6	14.7	0.41	6.4	5.4
10	14.9	4.3	3.84	5.6	12.2	0.33	5.2	4.0
<i>Totals</i>	861.3	106.4	<i>n.a.</i>	102.4	230.0	6.40	100	100

Thank you for your attention!

Discussion