

Quasi Risk-Neutral Pricing in Insurance

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ABSTRACT. This contribution shows that for certain classes of insurance risks, pricing can be based on expected values under a probability measure \mathbb{P}^* amounting to quasi risk-neutral pricing. This probability measure is unique and optimal in the sense of minimizing the relative entropy with respect to the actuarial probability measure \mathbb{P} , which is a common approach in the case of incomplete markets. After expounding the key elements of this theory, an application to a set of industrial property risks is developed, assuming that the severity of losses can be modeled by “Swiss Re Exposure Curves”, as discussed by Bernegger [1997]. These curves belong to a parametric family of distribution functions commonly used by pricing actuaries. The quasi risk-neutral pricing approach not only yields risk exposure specific premiums but also Risk Adjusted Capital (RAC) values on the very same level of granularity. By way of contrast, the conventional determination of RAC is typically limited to entire portfolios.

Keywords: incomplete markets, insurance prices, Lorenz order, property risks, quasi risk-neutral pricing, risk aversion, risk neutrality, risk management, Risk Adjusted Capital (RAC).

1. INTRODUCTION

The theory of pricing risk in incomplete markets based on risk-neutral valuation is well established [see e.g. Goovaerts et al., 1984, Delbaen and Haezendonck, 1989, or Gerber and Shiu, 1994]. While a risk-neutral probability measure is not unique in an

incomplete market, various approaches have been proposed and discussed for achieving minimum distance of the risk-neutral probability measure with respect to the actuarial measure [see e.g. Föllmer and Schweizer, 1991].

Coherent Risk measures, such as expected shortfall [see Artzner et al., 1999], are widely used in modern Risk Management, but relate to insurance portfolios rather than individual risks. On the contrary ad hoc pricing methods continue to be very much in use when it comes to pricing individual risks. This is surprising since distortion principles, which have been extensively discussed over the last decade in particular, provide a consistent approach to pricing single risks in general, and tail risk in particular [see e.g. Wang, 1995, 1997, or Denneberg, 1990, for some earlier work on distorted probabilities]. With respect to assets, Wang [2000] has bridged the gap between theory and practice by showing that the Black-Scholes formula can be reproduced by a distortion principle relating to Choquet pricing [for details on Choquet pricing see e.g. de Waegenaere et al., 2003].

The focus of this work is to motivate and apply Choquet pricing to industrial property insurance, which by the infrequent nature of loss events (excess of some basic threshold) is typically exposed to heavy tails. Although distortion principles do take into account the extra loading required for heavy-tailed loss distributions, even pricing actuaries familiar with Choquet pricing are hesitant to use them. One reason may be the considerable choice of distortion operators [see Wang, 1996, for an overview]. Once a distortion operator has been selected, the task of calibrating in a least arbitrary (and most sensible) way must still be solved.

In the present work we try to overcome these obstacles by proposing both

1. a distortion operator that is optimal in the above stated sense;
2. a unique rule of calibration based on Lorenz order [for a general overview see

Yaari, 1987].

A set of insurable risky prospects $\mathcal{X} \subset (\Omega, \mathcal{A}, \mathbb{P})$ and a pricing functional,

$$\begin{aligned} H : (\Omega, \mathcal{A}, \mathbb{P}) &\rightarrow \mathbb{R}^+ \\ X &\mapsto H(X), \end{aligned}$$

are assumed. Expected values under the actuarial probability measure \mathbb{P} are referred to as actuarial expectation or actuarially fair premium, implying a pricing rule in accordance with the concept of a long-term pure risk premium. We abstract from insurance cycles or price shocks caused by temporary shortages of insurance capacity. The pricing concept presented in this work takes into account the pure loss dynamics underlying an insurance portfolio, risk aversion of market participants and the insurer's solvency. However, it neglects administrative expense, which is very much company-specific.

This paper is structured as follows. Section 2 presents the axiomatic framework that provides the basis for the theoretical results reviewed in section 3 and the practical application to follow in section 4. After applying the theory of quasi risk-neutral pricing to the Swiss Re exposure curves in section 4.1, premiums and Risk Adjusted Capital values are calculated for a property insurance portfolio in section 4.2. Discussion and conclusions with an outlook to further research are given in the final sections 4.3 and 4.4. The Appendix offers some more detail of the pricing rule developed with respect to Swiss Re Exposure curves.

2. AXIOMATIC FRAMEWORK AND ESTABLISHED RESULTS

For ease of reference, we briefly revisit the set of axioms to be satisfied by H . With very little variation only these axioms have been extensively discussed in literature [see e.g. Goovaerts et al., 1984].

Axiom 1 [No Rip-off]. *For any risky prospect $X \in \mathcal{X}$, we require the pricing functional H to satisfy*

$$\mathbb{E}(X) \leq H(X) \leq \sup(X),$$

where

$$\sup(X) := \sup\{X(\omega) \mid \omega \in \Omega\}.$$

If $H(X) = \infty$, a risk is assumed non-insurable.

Axiom 2 [Monotonicity]. *The pricing functional H is required to preserve first-order stochastic dominance in that for any two risky prospects X_1 and X_2 , such that $X_1 \leq_{st} X_2$,*

$$X_1 \leq_{st} X_2 \implies H(X_1) \leq H(X_2).$$

The notation “ $X_1 \leq_{st} X_2$ ” means that X_2 exhibits first stochastic dominance over X_1 , which in terms of distribution functions implies that

$$F_{X_1}(x) \geq F_{X_2}(x), \forall x \in \mathbb{R}^+.$$

It is noted that this monotonicity assumption is weaker than the point-wise notion as discussed in Wang’s papers, since it just relates to distribution functions rather than to an underlying probability space. However, on any given probability space point-wise monotonicity implies monotonicity with respect to first stochastic dominance so that this axiom is neither unrelated nor in contrast to Wang’s monotonicity axiom.

Axiom 3 [Subadditivity]. *For any two risky prospects X_1 and X_2 it must hold true that*

$$H(X_1 + X_2) \leq H(X_1) + H(X_2).$$

This axiom mirrors the economic motivation of risk transfer, a fundamental principle of insurance.

Axiom 4 [Comonotonic Additivity]. *For any two comonotonic risky prospects X_1 and X_2 ,*

$$H(X_1 + X_2) = H(X_1) + H(X_2).$$

Comonotonic risks exhibit perfect positive stochastic dependence. For details concerning the concept of comonotonicity, see for instance Wang et al. [1997].

Axiom 5 [Continuity]. *For any increasing sequence of random prospects $X_n(\omega) \nearrow X(\omega)$, $\forall \omega \in \Omega$,*

$$\lim_{n \rightarrow \infty} H(X_n) = H(X).$$

For later reference, we recall the representation of H as integral. For some risk $X \in \mathcal{X}$ with a continuous distribution function on \mathbb{R}^+

$$H(X) = \int_{\mathbb{R}^+} x \pi'(F_X(x)) f_X(x) dx = \int_{\mathbb{R}^+} x d[\pi(F_X(x))], \quad (1)$$

where the distortion function π (π' denotes its derivative) mirrors the above axioms by the following properties,

($\pi 1$) π is increasing on $(0, 1)$;

($\pi 2$) π is convex on $(0, 1)$;

($\pi 3$) $\pi[0] = 0$ and $\pi[1] = 1$.

Details showing how the above axioms 1- 5 translate into these properties can for instance be found in Wang et al. [1997]. An additional property advocated by Wang [2000],

($\pi 4$) $\pi'[1] = \infty$,

deserves mentioning. It is of practical concern because it counterbalances the decreasing likelihood of large losses, preventing expected loss levels in high layers from approaching zero. Also, as shown in Wang [2000], there is a distortion operator having property $(\pi 4)$ which reproduces the Black-Scholes formula for option pricing. However, this property is not a logical consequence of the axiomatic framework introduced above. Hence, it will not be considered any further in this paper.

3. THE EXPONENTIAL DISTORTION OPERATOR

As stated in section 1, in a first motivating step for pricing actuaries to use Choquet pricing we propose a particular distortion operator which is optimal in the sense of minimizing the relative entropy with respect to the actuarial probability measure. This criterion results in a scaled exponential function

$$\pi^*(q) = \frac{1 - e^{\lambda q}}{1 - e^\lambda}, \quad 0 \leq q \leq 1, \quad \lambda \geq 0. \quad (2)$$

In order to assure convexity, as postulated by $(\pi 2)$, the parameter λ is non-negative. Details for obtaining (2), including a heuristic interpretation of λ as a risk-aversion parameter on the one hand and as Lagrange multiplier in a constrained optimization problem on the other hand, can be found in Niederau [2000, chapter 5 and Appendix B, respectively]. In a second step below, a calibration of (2) based on Lorenz order is developed. For details on Lorenz functions see for instance Johnson and Kotz [1995].

Proposition 1. *Two risks X_1 and X_2 with positive finite expectation are Lorenz ordered (notation: $X_1 \leq_L X_2$), if and only if*

$$L_{X_1}(p) \geq L_{X_2}(p), \quad \forall p \in (0, 1). \quad (3)$$

With respect to the unit interval the Lorenz function is defined as

$$L_X(p) = \frac{\mathbb{E}(F_X^{-1}(V) | V \leq p)}{\mathbb{E}(F_X^{-1}(V))}, \quad p \in [0, 1], \quad (4)$$

where $V \stackrel{d}{=} U(0, 1)$. One can also consider the Lorenz function on the image of X which leads to the expression

$$L_X(x) = \frac{\mathbb{E}(X | X \leq x)}{\mathbb{E}(X)}, \quad x \geq 0.$$

The latter representation is also denoted the first moment distribution of X . Some interesting aspects relating to Lorenz functions and their application in insurance is expounded for instance in Embrechts et al. [1997].

Theorem 1. *For any pair of comonotonic and partially Lorenz ordered random prospects X_1 and X_2 (stated in the same currency), with $\mathbb{E}(X_1) = \mathbb{E}(X_2) < \infty$ and $X_1 \leq_L X_2$, it holds true that*

$$\mathbb{E}(u(-X_1)) \geq \mathbb{E}(u(-X_2)) \iff H(X_1) \leq H(X_2), \quad (5)$$

for any increasing concave risk utility function u .

The “if” part in (5) is a consequence of the stop-loss order preserving property of H , as shown by Hürlimann [1998]. For the “only if” part, the reader is referred to the proof in Niederau [2000, Appendix E].

Remark 1. *This Theorem can be restated equivalently with respect to stop-loss order, i.e. second-order stochastic dominance, since in the presence of equal expectations, Lorenz and stop-loss order are equivalent as was originally shown by Hardy, Littlewood, and Polya in 1929 [for details see e.g. Shaked and Shantikumar, 1997].*

Remark 2. *Due to the latter result, equal expectations under \mathbb{P} and Lorenz order $X_1 \leq_L X_2$ guarantee that the left-hand side inequality of (5), which is assumed to hold for any increasing concave risk utility function u , is well defined. Likewise, note that for any $X \in \mathcal{X}$, $\mathbb{E}(X) = \infty \Rightarrow H(X) = \infty$, in which case a risk X was agreed to*

be non-insurable in axiom 1. In the case of the exponential distortion operator one even has $H(X) < \infty$ if and only if $\mathbb{E}(X) < \infty$. The latter equivalence motivates the assumption of finite means to assure that also the inequality on the right-hand side of (5) is well defined.

Remark 3. *Theorem 1 may appear restrictive due to the required equality in expectation. But even in presence of random prospects with unequal expected values (the usual case in reality), this apparent limitation can be overcome by using scaled random prospects. As can be verified [see e.g. Niederau, 2000, chapter 5], using the axiomatic framework revisited in section 2, the functional H is positively homogeneous of first degree, i.e. for any $X \in \mathcal{X}$ and $\alpha \in \mathbb{R}^+$, one has $H(\alpha X) = \alpha H(X)$. For scaling, let $\beta \equiv 1/\mathbb{E}(X)$ to find that $H(X) = \mathbb{E}(X) \cdot H(\beta X)$. Accordingly, with $\hat{X} := \beta X$, $H(X)$ is determined uniquely by $H(\hat{X})$, provided the actuarial expectation of X exists. Hence, in view of Theorem 1, it is sufficient to concentrate on scaled random prospects for the remaining part of this work.*

Remark 4. *The result of the Theorem can be extended by induction to any set of risky prospects which is totally Lorenz ordered. In the practical part of this paper we will encounter such a set which is characterized by the distribution functions of its elements (cf. remark 6, section 4.1).*

Remark 5. *While Theorem 1 is a general result which applies to any distortion operator, a unique rule of calibration with respect to the exponential distortion operator of (2) is developed in Niederau [2000, chapter 5, Lemma 1]. What is shown here is that (5) is equivalent to the existence of a bijective non-linear mapping,*

$$\begin{aligned} \lambda : [0.5, 1) &\rightarrow [0, \infty) \\ \xi &\mapsto \lambda(\xi), \end{aligned} \tag{6}$$

which is determined by the implicit equation

$$(e^\lambda - \lambda e^\lambda - 1) / (\lambda (e^\lambda - 1)) = \xi, \quad (7)$$

with ξ denoting the point on $[0, 1]$ at which the Lorenz function L_X associated with some underlying risk X assumes its mean value. Letting $V := F_X \stackrel{d}{=} U(0, 1)$, the very same ratio in (7) can be rewritten as

$$\mathbb{E}(V e^{\lambda V}) / \mathbb{E}(e^{\lambda V}). \quad (8)$$

To conclude this review of theoretical results, the pricing rule obtained from (7) in essence relates to the Esscher transform as discussed by Bühlmann [1980], according to (8). Here, however, the Esscher transform is applied to the distribution function F_X rather than to the random prospect X itself. The difference between these two types of transform is discussed e.g. in Wang [2000]. Note that Theorem 1 holds true for all increasing concave risk utility functions. Therefore, the equivalence in (5) means that all risk-averse (including risk-neutral) market actors order two such risks X_1 and X_2 as indicated by H . But since in the presence of comonotonic risks H has an expected value representation according to (1), a risk-averse market can be considered quasi risk-neutral with respect to the transformed density $f^*(x) := \pi'(F(x)) f(x)$. This notion of risk neutrality extends classical expected utility theory, where risk neutrality refers to a market where decision makers order risks uniquely with respect to actuarial expectation, implying linear risk utility functions. In this sense the qualifier “quasi” is used as a sign of caution rather than indicating a concept of risk neutrality beyond the theory of expected utility.

4. A PRACTICAL PRICING APPLICATION

When it comes to practical application, there are many ad hoc methods to calibrate the distortion operator in (2). For instance, one way is to fix a layer premium for some

basic layer (e.g. the first million Euro) and calibrate λ to match a preset premium for that basic layer. This value of λ then determines the risk loading for any higher layer. However, this approach is judgemental in various respects. Even choosing a "meaningful" basic layer and its premium, assumes that the loss dynamics in the basic layer are indicative for the pricing of high-excess layers. The latter approach characterizes layer pricing with increased limit factors (ilf) [see for details Wang, 1995].

By way of contrast, a consistent rule of calibration is developed in section 4.2 below, using Theorem 1 and its extension as alluded to in remark 4. Rather than relying on some layer for calibrating λ , this rule takes into account the shape of the whole distribution function by means of Lorenz order. In section 4.2, this theory will be applied to an assumed wind risk portfolio such as typically insured by industrial insurers. A member of the Maxwell-Boltzmann, Boese-Einstein, Fermini-Dirac (MBBEFD for short) class of distribution functions as discussed by Bernegger [1997] will be used to model the severity of loss in such a portfolio.

4.1. Swiss Re exposure curves and Lorenz order . The MBBEFD function of most practical interest for insurance applications [cf. Bernegger, 1997 for an extensive discussion] is given by

$$\sigma(x) = \frac{\ln\left(\frac{(g-1)b+(1-gb)b^x}{1-b}\right)}{\ln(gb)}, \quad (9)$$

$b > 0, b \neq 1, bg \neq 1, g > 1, x \in [0, 1]$, where with $c > 0$

$$\begin{aligned} b &:= \exp\{3.1 - 0.15c(1+c)\}, \\ g &:= \exp\{c(0.78 + 0.12c)\}. \end{aligned} \quad (10)$$

The underlying distribution function is given by

$$F_{b,g}(x) = 1 - \frac{1-b}{(g-1)b^{1-x} + (1-gb)}. \quad (11)$$

In practical insurance pricing the ensemble (9) and (10) is referred to as the Swiss Re Exposure curve. Subject to the calibration in (10), which will also be assumed below, in terms of shape both functions (9) and (11) are completely characterized by the value of c . A risk X with distribution function (11) is referred to as being of type $sre(c)$ below. Note that the functions (9) and (11) are not defined for the critical value

$$\tilde{c} = -0.5 + \sqrt{0.25 + 3.1/0.15} \approx 4.07, \quad (12)$$

where they have a different representation [see Bernegger, 1997, for details].

Some important facts and interpretations are outlined below while again, for the full detail, the reader is referred to Bernegger's work.

1. In both (9) and (11), $x = y/MPL$, where y relates to possible realizations of some underlying random prospect Y which is bound from above by its Maximum Possible Loss (MPL) which is assumed finite in property insurance. Note that in insurance terminology MPL does not necessarily mean total physical destruction; rather it stands for maximum damage if all risk protection and prevention measures fail. The calibration given in (10) reproduces certain pricing schemes used in industrial reinsurance markets. In particular, for $c = 5$ one obtains the so-called Y6 curve, a premium allocation scheme used by Lloyd's and many other industrial insurers.
2. If Y is of type $sre(c)$, then

$$\mathbb{E}(Y) = \frac{MPL_Y \ln(gb)(1-b)}{\ln(b)(1-gb)},$$

the factor

$$\frac{\ln(gb)(1-b)}{\ln(b)(1-gb)} \quad (13)$$

just being the expectation of $X = Y/MPL_Y$. Unlike the c -parameter, the MPL thus serves as a scale parameter.

3. if Y is of type $sre(c)$, its limited expected value function (LEV) is

$$LEV_Y(y) := \mathbb{E}(Y|Y \leq y) = \sigma_Y \left(\frac{y}{MPL_Y} \right) \mathbb{E}(Y), \quad y \geq 0. \quad (14)$$

Proposition 2. *Let Y_1 and Y_2 be $sre(c_1)$ and $sre(c_2)$, respectively with $c_1, c_2 > 0$. Then Y_1 and Y_2 are Lorenz ordered.*

Proof. We assume $\mathbb{E}(Y_1) = \mathbb{E}(Y_2)$ which is justified by the scale invariance property of Lorenz order. Recalling the second property above, this equality in expectations implies a parametric constraint given by

$$\frac{\ln(g_1 b_1) (1 - b_1) \ln(b_2) (1 - g_2 b_2)}{\ln(b_1) (1 - g_1 b_1) \ln(g_2 b_2) (1 - b_2)} = \frac{MPL_2}{MPL_1}, \quad (15)$$

where the indices 1 and 2 relate to Y_1 and Y_2 , respectively. Now it needs to be shown that either $Y_1 \leq_{sl} Y_2$, or $Y_1 \geq_{sl} Y_2$, where the symbol “ \leq_{sl} ” denotes stop-loss order. This will already conclude the proof according to remark 1 in section 3. Assume $c_1 \leq c_2$, then along the contour given by (15) one has

$$\sigma(x_1|c_1) \geq \sigma(x_2|c_2),$$

with $x_i := y/MPL_i$, $i = 1, 2$ and $y \geq 0$. But due to the third property above, for all $y \geq 0$ this is equivalent to

$$\begin{aligned} LEV_1(y) &\geq LEV_2(y), \\ \iff \mathbb{E}(Y_1) - LEV_1(y) &\leq \mathbb{E}(Y_2) - LEV_2(y) \\ \iff \mathbb{E}(Y_1|Y_1 \geq y) &\leq \mathbb{E}(Y_2|Y_2 \geq y) \\ \iff Y_1 &\leq_{sl} Y_2. \end{aligned}$$

The latter equivalence is shown for instance by Antal [1997] and finishes the proof.

Remark 6. *The proof highlights that for risks of type $sre(c)$, Lorenz order is determined by the order of the according values of c . Moreover, since c_1 and c_2 were chosen*

arbitrarily the set of all risks which are of type $sre(c)$ is even totally ordered under the relation \leq_L (cf. remark 4 in section 3) which motivates the application as outlined in the next section.

4.2. Calculation of prices and RAC in a quasi risk-neutral setting . The data shown in table 1 below assume a selection of industrial storm exposures in Belgium and the Netherlands. For the sake of focus, actuarial parameters in this case study,

TABLE 1 OVERVIEW OF EXPOSURE DATA AND PRICING RESULTS.

Loc	Ins Val	MPL	c	$\mathbb{E}(Z_i)$	$H(Z_i)$	RAC	RAC *	RAC ⁺
1	261.9	26.6	4.31	21.8	49.4	1.38	21.4	25.0
2	233.7	19.3	4.26	16.5	37.3	1.04	16.3	18.1
3	130.2	13.2	4.17	12.4	27.8	0.77	12.0	12.4
4	64.1	8.8	4.08	9.1	20.4	0.57	8.9	8.2
5	47.8	7.9	4.05	8.4	18.7	0.52	8.1	7.5
6	36.7	7.3	4.04	7.9	17.6	0.49	7.7	6.9
7	29.9	6.9	4.03	7.5	16.7	0.46	7.2	6.5
8	22.8	6.4	4.00	7.1	15.7	0.43	6.7	6.0
9	19.3	5.7	3.96	6.6	14.7	0.41	6.4	5.4
10	14.9	4.3	3.84	5.6	12.2	0.33	5.2	4.0
<i>Totals</i>	861.3	106.4	<i>n.a.</i>	102.4	230.0	6.40	100	100

i.e. those relating to the severity and annual frequency of storm losses, are assumed to be given best-estimates and will not be discussed in any detail. The labels, “Loc” and “Ins Val” denote the location and the insured value, respectively. Throughout this section, sums insured, MPL and RAC values are stated in million (mn.) of Euros, while

(fair) premium values are stated in thousands of Euros. Due to the typical pathways of European cyclones, locations in Belgium and the Netherlands are usually exposed to the same storms, causing them to present comonotonic risks. For all locations the value of c differs from the critical value of 4.07, as in (12), indicating that the distribution function in (11) is well defined. Therefore the data satisfy all prerequisites of Theorem 1 and remark 4, since (besides comonotonicity) these ten locations are completely Lorenz ordered by their value of c , according to Proposition 2. A full value cover is assumed for the sake of simplicity, meaning that neither the insured nor the insurer impose any loss limits other than the location-specific MPL¹. A common expected loss frequency of 3.5% percent per location and per annum was assumed. In such a situation theory suggests using a Poisson distribution. However, from a practical viewpoint, in the presence of such a low frequency the probability of more than one loss occurring per year is negligibly low (an argument which is elaborated in some more detail by Daykin, et al. [1994] in the context of so-called shadow claims). This is why in this practical study a diatomic distribution with parameters $p_i = 0.035$ and $n_i = 1, \forall i = 1, \dots, 10$, respectively, is used for simplicity. The diatomic representation of the loss frequency implies that with probability p_i one loss occurs at location i and accordingly, no loss occurs with the counter probability $1 - p_i$. Thus, treating each location as a single risk unit X_i , the total portfolio loss can be written as

$$Z = \sum_{i=1}^{10} Z_i = \sum_{i=1}^{10} \mathbb{1}_{\{N=1\}} X_i, \tag{16}$$

where

$$H(Z) = \sum_{i=1}^{10} [p_i H(X_i) + (1 - p_i) H(0)] = \sum_{i=1}^{10} \left[p_i H(\hat{X}_i) \mathbb{E}(X_i) \right]. \tag{17}$$

¹This is clearly an abstraction from reality since insurers would usually sublimit their exposures to natural perils in order to counteract loss accumulation.

In accordance with the notation introduced before, \hat{X}_i refers to the scaled loss variable $X_i/E(X_i)$. Here, the pricing functional H takes into account that no loss occurs with probability $1 - p_i$, which does not give rise to any premium in keeping with Axiom 1. The total expected loss per location is obtained by Wald's identity, i.e. $E(Z_i) = p_i E(X_i)$. Recalling (17), $H(Z_i)$ is nothing but $p_i H(X_i)$, which is why

$$H(Z_i)/\mathbb{E}(Z_i) = H(X_i)/\mathbb{E}(X_i).$$

The latter ratio can thus be interpreted as the loading factor charged for location i [see table 3 in the Appendix for a set of loading factors as a function of c].

The determination of the (one-year) Risk Adjusted Capital (RAC), or more precisely, the return on Risk Adjusted Capital (RoRAC), is an important issue to insurers, since (in conjunction with the insurer's β from the CAPM) it helps investors to position an insurer relative to their efficient portfolio frontier [see for details e.g. Zweifel and Auckenthaler, 2007]. Now recall that under quasi risk-neutrality, the location-specific surcharges $H(Z_i) - E(Z_i)$ can be interpreted as certainty-equivalent net cash-flows. Assuming discrete compounding for simplicity, this permits to uniquely determine the respective present value of RAC as

$$RAC_i = [H(Z_i) - \mathbb{E}(Z_i)]/r_f, \quad (18)$$

where r_f denotes the one-year risk-free hurdle rate. The according RAC values are shown in table 1 for an assumed hurdle rate of 2 percent per annum. Total RAC, being defined as

$$RAC_{tot} = \sum_{i=1}^{10} RAC_i,$$

amounts to about 6 percent of total MPL, or Euro 6.4 mn. Table 2 shows that this value is close to the 99.6% quantile of the aggregate loss distribution of the portfolio risk as given in (16). This implies that only once in 250 years will the aggregate loss

TABLE 2 SELECTED PERCENTILES OF THE AGGREGATE PORTFOLIO DISTRIBUTION.

loss amount	percentile
0	96.50
0.71	97.00
1.08	97.50
1.47	98.00
1.98	98.50
3.32	99.00
5.89	99.50
6.45	99.60
8.54	99.75
12.11	99.90

burden exceed RAC_{tot} , forcing the insurer to raise capital beyond this value.

It may be worthwhile to emphasize the sensitivity of these estimates to the cutoff point chosen. At the 95% quantile, no RAC would have been necessary, while at 99% quantile (a preferred choice in practical applications), RAC would amount to some Euro 3.2 mn., just about one-half of the calculated value [see table 2]. More generally, quantile-based rules for the determination of RAC may expose an insurer to a great potential for error, particularly in the presence of highly skewed loss distributions. However, even when using the exact RAC calculation rule, insurers may still be concerned about parameter uncertainty, a topic which is not addressed in this paper but merits mentioning [for details on the treatment of parameter uncertainty see e.g. Wang, 2003]. In table 1, the allocation as derived from the quasi risk-neutral pricing

(indicated by RAC^* , in percent of RAC_{tot}), is juxtaposed to a typical practitioner's rule-of-thumb allocation RAC^+ , in percent of the very same amount RAC_{tot} . The rule of thumb allocates RAC according to the location's relative MPL, such that

$$\frac{RAC_i^+}{RAC_{tot}} = \frac{MPL_i}{\sum_{i=1}^{10} MPL_i}.$$

As can be seen from the last column of table 1, this rule biases RAC allocation in favor of the locations with high MPL values. By way of comparison, the allocation based on quasi risk-neutral pricing results in a more balanced RAC allocation. Since RAC is often used not only as a measure of risk tolerance but also as a means to measure underwriting performance, choosing an appropriate allocation rule is of considerable importance. The rule based on quasi risk-neutral pricing can be argued to be preferable because it takes account of the whole distribution function of the risks involved rather than just focussing on the maximum (foreseeable) exposure to loss, a mere reflection of risk aversion. In the present context it recognizes the fact that losses in property insurance do not necessarily occur at locations with large MPL but also hit locations of medium to small size.

4.3. Discussion. While providing interesting insights, this case study cannot claim general validity. Usually, insurance portfolios are neither comonotonic nor are they completely Lorenz ordered. However, Theorem 1 can be generalized beyond comonotonicity and Lorenz order in the following sense. Denote with X_1 and X_2 two single risks (e.g. two locations as in the practical application). Then axiom 3 implies

$$H(X_1 + X_2) \leq H(X_1) + H(X_2).$$

In turn, provided X_1 and X_2 have finite expectation, $H(X_1)$ and $H(X_2)$ are uniquely determined by $H(\hat{X}_1)$ and $H(\hat{X}_2)$. The latter quantities can be uniquely calculated

following the logic of Theorem 1 and the calibration motivated in remark 5, since X_1 and X_2 are both Lorenz ordered and comonotonic with respect to themselves.

These considerations can be used to calculate at least an upper bound RAC for any insurance portfolio. If $PF^{(I)} = (Y_i)_{i=1,\dots,n}$ denotes some insurance portfolio, Y_i single risks for all i , and $RAC(PF^{(I)})$ the one-year RAC for this portfolio, then the theory predicts

$$RAC(PF^{(I)}) \leq \sum_{i=1}^n [H(Y_i) - \mathbb{E}(Y_i)] / r_f \tag{19}$$

This inequality is of interest for at least three reasons. First, from a risk management point of view, it provides a value of the maximum RAC assuming a complete lack of diversification effects in the portfolio under consideration. Second, the ratio

$$\frac{r_f RAC(PF^{(I)})}{\sum_{i=1}^n [H(Y_i) - \mathbb{E}(Y_i)]} \leq 1 \tag{20}$$

may serve as an operational measure of diversification effects. Indeed, (20) can be interpreted as the benefit of diversification to potential purchasers of insurance. The closer the ratio on the left-hand side of (20) is to zero the higher the benefit to the insured is. Third, from a shareholder value point of view, excessive RAC, indicated by a violation of (20), is a cause of concern. Too much costly capital would be tied up by underwriting and as a consequence, return to capital is both understated and lower than necessary, to the detriment of shareholders. However, this means that an insurance company runs the risk of not being on investors' efficient frontier in terms of expected returns and volatility of returns, unless this shortfall can be made up by a success in capital investments that outperforms the other investors in the capital market [see e.g. Zweifel and Auckenthaler, 2007].

4.4. Conclusions and outlook. The objective of this contribution is to derive a quasi risk-neutral pricing rule for insurance that amounts to an expected value, defined

under a modified probability measure. While assuring minimum relative entropy with respect to the actuarial probability measure, this modification reflects risk aversion of market actors in the insurance industry. To the extent that insurance risks are comonotonic and exhibit Lorenz order, loading factors such as mirrored in the insurer's Risk Adjusted Capital can be determined uniquely. They call for a higher surcharge to those (scaled) risks dominating under Lorenz order. Moreover, a maximum price can be derived for any portfolio by abstracting from diversification effects, i.e. by pricing every single risk unit independently. Quasi risk-neutral pricing is applied to a set of industrial plants having exposure to storm loss. Location-specific RAC values are calculated and compared to a typical practitioner's rule that allocates RAC according to the location's maximum possible loss (MPL) in proportion to the portfolio's total MPL. Since quasi risk-neutral pricing is based not only on the maximum exposure to loss but also takes into account the shape of the entire loss distribution, it avoids underreserving for risks that have considerable loss potential in the small to medium range which fails to be reflected in their limited MPL values.

In view of further research, the maximum price derived for any portfolio such as motivated in this work may be refined to yield more accurate upper bounds (or even exact values) by including the effects of risk diversification. In the same vein, insurers' capital investments should be taken into account because they provide additional hedging opportunities. These considerations might result in a more general rule for RAC allocation, with RAC determined by a coherent risk measure in the sense of Artzner et al. [1999].

APPENDIX

As show in table 3, the sensitivity of the loading factors is highest for values of c up to about 3.5. For higher values of c , such as considered in section 4.2, they

still increase but at a decreasing rate. These values may appear counter-intuitive to insurance practitioners who would usually consider a risk characterized by $c = 1$ as being more risky than one with $c = 5$ (e.g.) because the former exhibits more exposure to MPL and hence seems to require more risk capital. This apparent paradox can

TABLE 3 LOADING FACTORS AND RELATED PARAMETERS AS A FUNCTION OF THE c -VALUE

c-value	ξ	$\lambda(\xi)$	loading factor
1.0	0.5784	0.955	1.21
2.0	0.6199	1.492	1.45
2.5	0.6335	1.676	1.62
3.0	0.6644	2.115	1.97
3.5	0.6771	2.307	2.12
4.0	0.6856	2.439	2.22
4.5	0.6905	2.517	2.28
5.0	0.6932	2.560	2.33
5.5	0.6946	2.583	2.37
6.0	0.6953	2.595	2.39
6.5	0.6957	2.601	2.40
7.0	0.6960	2.606	2.41

be explained by recalling the separation of scale and risk, mentioned in remark 3 in section 3. According to the parametrization in (7), when c approaches zero this means approaching the deterministic case, i.e. MPL being realized with certainty². While depending on the size of the MPL, such an exposure to loss can call for a great deal

²Amunition plants or any industrial plant exposed to vapor cloud explosion may serve as an example.

of capital. However, in keeping with axiom 1, this capital will be completely provided by the premium charged, which in this case just coincides with the MPL and therefore with the actuarially fair premium. Hence, in the limit, when c approaches zero, there is no need or even justification (according to the "no rip-off" part of axiom 1) to charge a premium beyond the actuarially fair value. Conversely, a risk with $c = 5$ exhibits quite a skewed loss distribution and the actuarially fair premium is but 1.22 percent of the MPL, regardless of its value³. In case of such a risk, the insurer needs to raise extra risk capital in order to fund a potential MPL. The associated opportunity cost of risk capital is reflected in the loading factor of 2.33, as shown in table 3. A set of comparative loading factors for the generalized Pareto distribution, discussed for instance in Embrechts et al. [1997], can be found in Niederau [2000, Appendix C].

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³This value can be verified using formula (13).

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